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Melvin twists of global $AdS_5 \times S_5$ and their non-commutative field theory dual

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Abstract: We consider the Melvin Twist of $AdS_5 \times S_5$ under $\mathrm{U}(1) \times \mathrm{U}(1)$ isometry of the boundary S_3 of the global AdS_5 geometry and identify its field theory dual. We also study the thermodynamics of the Melvin deformed theory.

Keywords: AdS-CFT Correspondence, Non-Commutative Geometry.

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Type of Twist	Model
Melvin Twist	Hashimoto-Thomas model
Melvin Shift Twist	Seiberg-Witten Model
Null Melvin Shift Twist	Aharony-Gomis-Mehen model
Null Melvin Twist	Dolan-Nappi model
Melvin Null Twist	Hashimoto-Sethi model
Melvin R Twist	Bergman-Ganor model
Null Melvin R Twist	Ganor-Varadarajan model
R Melvin R Twist	Lunin-Maldacena model

Table 1: Catalog of non-commutative gauge theories viewed as a world volume theory of D-branes in a "X" Melvin "Y" twist background. This table originally appeared in [18].

Melvin twist, also known as the T-s-T transformation, is a powerful solution generating technique in supergravity and string theories [1 − 6]. The procedure relies on having a $U(1) \times U(1)$ compact isometry along which one performs a sequence of T-duality, twist, and a T-duality. The twist is an $SL(2, R)$ transformation on the complex structure of the T-dual torus. As such, the Melvin twist can simply be thought of as an $SL(2, R)$ transformation acting on the Kähler structure of the torus parameterized by $U(1) \times U(1)$.

Interesting closed string backgrounds, such as Melvin universes, null branes, pp-waves, and Gödel universes can be constructed by applying the Melvin Twist procedure to the Minkowski background. The construction reveals the hidden simplicity of these closed string backgrounds: they are dual to flat spaces. As a result, world sheet sigma model for strings in these backgrounds are exactly solvable and have been studied extensively [7 − 12]. The same procedure can be applied to black p-brane backgrounds to construct various asymptotically non-trivial space-time geometries [13].

Melvin twist applied to the Dp-brane background and the subsequent near horizon limit gives rise to supergravity duals for a variety of decoupled field theories[1] depending on the orientation of the brane and the Melvin twist. If both of the $U(1)$ isometries are along the brane, one generally obtains a non-commutative field theory, typically with non-constant non-commutativity parameter [14 − 19]. If one of the $U(1)$ is transverse to the brane, then one obtains a dipole field theory [20 − 22]. Taking both of the $U(1)$'s to be transverse to the brane gives rise to the construction of Lunin and Maldacena [23]. The list of models constructed along these lines is summarized in table 1. These theories are S-dual to NCOS theories [24, 25]. They are also closely related to "Puff Field Theory" which was studied recently in [26, 27]. The hidden simplicity of Melvin twists in the context of gauge theory duals manifests itself as preservation of integrability. The fact that q/β-deformed $\mathcal{N} = 4$ SYM remains integrable was pointed out in [28, 29]. A broader class of integrable twists were studied in [30, 31].

In this article, we consider the effect of twisting along the $U(1) \times U(1) \in SO(4)$ isometry

[1] An earlier discussion of a construction of this type is [6].

of the S_3. More specifically, we consider $AdS_5 \times S_5$ solution of type IIB theory

$$\begin{aligned} ds^2 &= R^2 \left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho (d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2) + d\Omega_5^2 \right] \\ B &= 0 \\ e^\phi &= \frac{\lambda}{4\pi N} \end{aligned} \tag{1}$$

where λ is the 't Hooft coupling

$$\lambda = 2g_{\rm YM}^2 N = 4\pi g_s N = \frac{R^4}{\alpha'^2}\,, \tag{2}$$

and perform a Melvin twist on the torus parameterized by the coordinates (ϕ_1, ϕ_2). This is equivalent to acting on the Kahler structure

$$\rho = \frac{1}{\alpha'}\left(B_{\phi_1\phi_2} + i\sqrt{g_{\phi_1\phi_1} g_{\phi_2\phi_2}}\right) \tag{3}$$

by an SL$(2, R)$ transformation

$$\rho \to \rho' = \frac{\rho}{\chi\rho + 1} \tag{4}$$

giving rise to a background

$$\begin{aligned} ds^2 &= \alpha'\sqrt{\lambda}\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho \left(d\theta^2 + \frac{\sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2}{1+\chi^2\lambda\cos^2\theta\sin^2\theta\sinh^4\rho}\right) + d\Omega_5^2 \right] \\ B &= \alpha' \left(\frac{\lambda\chi\cos^2\theta\sin^2\theta\sinh^4\rho}{1+\chi^2\lambda\cos^2\theta\sin^2\theta\sinh^4\rho}\right) d\phi_1 \wedge d\phi_2 \\ e^\phi &= \left(\frac{1}{\sqrt{1+\chi^2\lambda\cos^2\theta\sin^2\theta\sinh^4\rho}} \right) \frac{\lambda}{4\pi N} \end{aligned} \tag{5}$$

with suitable Ramond-Ramond fields. This is a deformation of the $AdS_5 \times S_5$ geometry (1) with respect to single dimensionless parameter χ. The $AdS_5 \times S_5$ geometry is recovered in the limit $\chi \to 0$. The goal of this article is to identify the interpretation of the deformation with respect to χ on the field theory side of the AdS/CFT correspondence.

Precisely the deformation of this type was studied in [30], and as these authors suggested, it is quite natural to interpret this background as being dual to a non-commutative deformation of $\mathcal{N} = 4$ SYM on $R \times S_3$ with the Moyal $*$-product

$$f * g = e^{2\pi i\chi\left(\frac{\partial}{\partial\phi_1}\frac{\partial}{\partial\phi_2'} - \frac{\partial}{\partial\phi_2}\frac{\partial}{\partial\phi_1'}\right)/2} f(\tau,\theta,\phi_1,\phi_2) g(\tau,\theta,\phi_1',\phi_2')\Big|_{\phi_1=\phi_1',\phi_2=\phi_2'}\,. \tag{6}$$

This interpretation fits naturally with the established patterns seen in other non-commutative field theories [14−19]. The naturalness of this interpretation is also echoed in [32].

There is however a problem in making this identification more precise. The gauge/gravity dualities are motivated by the complementarity of black D3-branes of string

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theory in various regimes of the t'Hooft coupling λ [33]. This allowed for an explicit analysis of the physics of open string degrees of freedom, which gave rise to a concrete realization of non-commutative dynamics in the appropriate decoupling limit. The $\mathrm{U}(1) \times \mathrm{U}(1)$ isometry which we exploited in constructing the χ deformation is an isometry of the near horizon $AdS_5 \times S_5$ geometry but not of the full D3-brane geometry. This makes the direct analysis of the open string dynamics from the world sheet point of view along the lines of [34] impossible.

We will show in this article that embedding into full D3 geometry is still possible, by exploiting the underlying SL$(2, Z)$ T-duality structure of the (ϕ_1, ϕ_2) torus. This is the string theoretical manifestation of the Morita equivalence in non-commutative field theories. To take advantage of this duality, it is useful to restrict to the case where χ is a rational number. Then, there exists an SL$(2, Z)$ transformation which removes the non-locality. Since this SL$(2, Z)$ dual is a local theory, it is the description most suitable for exploring the deep UV behavior [35]. The SL$(2, Z)$ structure in fact gives rise to a self-similar phase diagram similar to the fundamental domain of the moduli-space of a torus. Similar structures have been shown to arise in NCOS [36] and PFT [27] theories as well. Since rational numbers are dense, this will suffice for the purpose of identifying the field theory dual of (5). In other words, we can use the fact that the effective theory in the IR region of the phase diagram depends smoothly on χ.

Let us suppose, for sake of concreteness, that

$$\chi = \frac{s}{p} \tag{7}$$

for relatively prime integers p and s. Then, one can find integers r and q so that

$$\begin{pmatrix} r & q \\ -s & p \end{pmatrix} \in \mathrm{SL}(2, Z) \ . \tag{8}$$

Acting on the Kahler structure ρ' for the background (5) by this SL$(2, Z)$ transformation gives rise to

$$\rho'' = \frac{r\rho' + q}{-s\rho' + p} = \frac{q}{p} + \frac{i}{p^2}\sqrt{\lambda}\cos\theta\sin\theta\sinh^2\rho \ . \tag{9}$$

In other words, the supergravity background is transformed to take the form

$$\begin{aligned} ds^2 &= \alpha'\sqrt{\lambda}\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \frac{\sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2}{p^2}\right) + d\Omega_5^2\right] \\ B &= \alpha'\frac{q}{p}d\phi_1 \wedge d\phi_2 \\ e^\phi &= \frac{1}{p^2}\frac{\lambda}{4\pi N} \end{aligned} \tag{10}$$

where ϕ_1 and ϕ_2 are periodic with respect to 2π. We can change variables

$$\phi_i = p\tilde{\phi}_i, \qquad i = 1, 2 \tag{11}$$

and write

$$\begin{aligned} ds^2 &= \alpha'\sqrt{\lambda}\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\tilde{\phi}_1^2 + \cos^2\theta d\tilde{\phi}_2^2\right) + d\Omega_5^2\right] \\ B &= \alpha' qpd\tilde{\phi}_1 \wedge d\tilde{\phi}_2 \\ e^{\phi} &= \frac{1}{p^2}\frac{\lambda}{4\pi N} \end{aligned} \tag{12}$$

with

$$\tilde{\phi}_i \sim \tilde{\phi}_i + \frac{2\pi}{p}, \qquad i = 1, 2\ . \tag{13}$$

This solution is therefore recognizable as a $Z_p \times Z_p$ orbifold of $AdS_5 \times S_5$ with pN units of RR-flux threading the S_5. This type of orbifold, acting on the AdS_5 sector of the geometry, was first considered in [37]. Now, this solution is no less easier to embed in the full D3 solution for its dynamics to be interpreted from the open string point of view than (5), because of the orbifolding with respect to the killing vectors

$$\xi_i = \frac{\partial}{\partial\tilde{\phi}_i}, \qquad i = 1, 2\ . \tag{14}$$

However, its covering space is simply $AdS_5 \times S_5$ with some exact B field. *This* is easier to embed into the D3 geometry.

In order to explore the embedding into the full D3 geometry, it is convenient to first go to the Poincare coordinate of the $AdS_5 \times S_5$ geometry. This can be accomplished by recalling the two different ways of parameterizing the hyperboloid

$$\begin{aligned} \frac{R}{2u}(1 + u^2(R^2 + x_1^2 + x_2^2 + x_3^2 - t^2) &= X_0 = R\cosh\rho\cos\tau \\ Rux_1 &= X_1 = R\sinh\rho\sin\theta\cos\tilde{\phi}_1 \\ Rux_2 &= X_2 = R\sinh\rho\sin\theta\sin\tilde{\phi}_1 \\ Rux_3 &= X_3 = R\sinh\rho\cos\theta\sin\tilde{\phi}_2 \\ \frac{R}{2u}(1 - u^2(R^2 - x_1^2 - x_2^2 - x_3^2 + t^2)) &= X_4 = R\sinh\rho\cos\theta\cos\tilde{\phi}_2 \\ Rut &= X_5 = R\cosh\rho\sin\tau \end{aligned} \tag{15}$$

satisfying $X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 + X_5^2 = R^2$ in $R^{2,4}$.

This implies a map between coordinates

$$\begin{aligned} \tilde{\phi}_1 &= \arg\left(x_1 + ix_2\right) \\ \tilde{\phi}_2 &= \arg\left(\frac{\left(-R^2 - t^2 + x_1^2 + x_2^2 + x_3^2\right)u^2 + 1}{2} + iRu^2x_3\right) \\ \theta &= \arg\left(\sqrt{R^2u^2x_3^2 + \frac{\left(u^2\left(R^2 + t^2 - x_1^2 - x_2^2 - x_3^2\right) - 1\right)^2}{4u^2}} + iRu\sqrt{x_1^2 + x_2^2}\right) \end{aligned}$$

$$\begin{aligned}\tau &= \arg\left(\frac{\left(R^2 - t^2 + x_1^2 + x_2^2 + x_3^2\right)u^2 + 1}{2} + iRtu^2\right)\\ \rho &= \cosh^{-1}\left(\sqrt{t^2u^2 + \frac{\left(\left(R^2 - t^2 + x_1^2 + x_2^2 + x_3^2\right)u^2 + 1\right)^2}{4u^2R^2}}\right) \ .\end{aligned} \tag{16}$$

In terms of the Poincare coordinates, the supergravity background takes on a simple form

$$ds^2 = R^2\left(u^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2\right) \tag{17}$$

and the B-field having the form

$$B = \alpha' qp \frac{\partial\tilde{\phi}_1}{\partial x_\mu}\frac{\partial\tilde{\phi}_2}{\partial x_\nu}\, dx^\mu \wedge dx^\nu \ . \tag{18}$$

The fact that $dB = 0$ ensures that the $AdS_5 \times S_5$ solution is unperturbed. Suppose we rescale

$$u = \frac{r}{R^2} \tag{19}$$

which makes the metric take the form

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\left(\frac{dr^2}{r^2} + d\Omega_5^2\right) \ . \tag{20}$$

It is then possible to extend this solution to full D3

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{21}$$

while continuing to let the B-field have the form (18) which continues *not* to back react.

In the large r limit, B becomes

$$B = \alpha' qp\, d\tilde{\phi}_1 \wedge d\tilde{\phi}_2 \tag{22}$$

where

$$\tilde{\phi}_1 = \arg(x_1 + ix_2), \qquad \tilde{\phi}_2 = \arg(-R^2 - t^2 + x_1^2 + x_2^2 + x_3^2) \ . \tag{23}$$

What this suggests is that the covering space of (12) is interpretable as $\mathcal{N} = 4$ gauge theory with background field

$$F = \frac{B}{\alpha'} = qp\, d\tilde{\phi}_1 \wedge d\tilde{\phi}_2 \tag{24}$$

in the decoupling limit. It is straight forward to verify that the equations of motion and the Bianchi identity for the gauge fields

$$d * F = 0 = dF \tag{25}$$

are satisfied. However, since the flux is fractional, it must be interpreted as giving rise to a 't Hooft flux [38].

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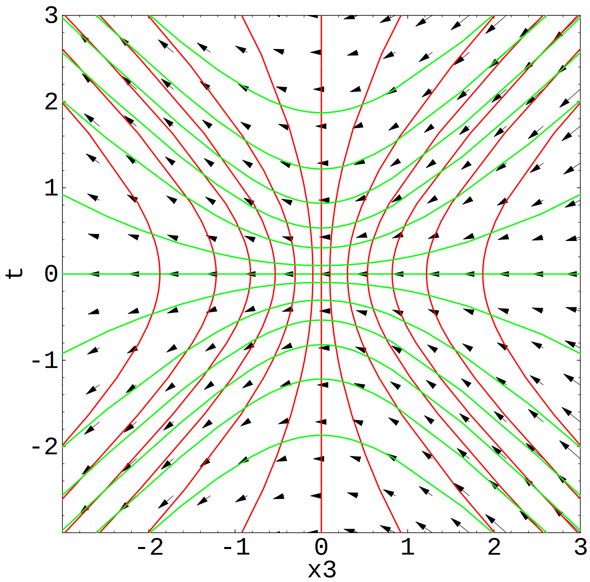

Figure 1: The contour of fixed τ (green) and fixed $\tilde{\phi}_2$ (red) in the $\theta = 0$ hypersurface which amounts to setting $x_1 = x_2 = 0$. The arrows represent the field of Killing vector ξ_2.

Our remaining task in addressing our original motivation is to work out the implication of (24) in identifying the field theory dual of (5). To facilitate this, it is useful to first work out the map which relates the coordinates on the boundary of global AdS_5 to the the boundary of Poincare AdS_5. This is achieved by taking the large u limit of (16) which reads

$$\begin{aligned}
\tilde{\phi}_1 &= \arg\left(x_1 + ix_2\right) \\
\tilde{\phi}_2 &= \arg\left(\frac{-R^2 - t^2 + x_1^2 + x_2^2 + x_3^2}{2} + iRx_3\right) \\
\theta &= \arg\left(\sqrt{R^2 x_3^2 + \frac{\left(R^2 + t^2 - x_1^2 - x_2^2 - x_3^2\right)^2}{4}} + iR\sqrt{x_1^2 + x_2^2}\right) \\
\tau &= \arg\left(\frac{R^2 - t^2 + x_1^2 + x_2^2 + x_3^2}{2} + iRt\right) \ .
\end{aligned} \tag{26}$$

Since we will ultimately compactify along the isometry vectors (14), it would be instructive to see how these vectors are oriented in the Poincare coordinates. We illustrate in figure 1 the contour of fixed τ and fixed $\tilde{\phi}_2$ in the $\theta = 0$ hypersurface which amounts to setting $x_1 = x_2 = 0$.

It is also useful to specify the metric for the space on which the field theory is defined. Starting with the round metric on $R \times S_3$

$$ds^2 = R^2\left[d\tau^2 + d\theta^2 + \sin^2\theta d\tilde{\phi}_1^2 + \cos^2\theta d\tilde{\phi}_2^2\right] \tag{27}$$

and applying (26) maps this to a conformally flat metric

$$ds^2 = f(t, x_1, x_2, x_3)(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) \tag{28}$$

with

$$f(t, x_1, x_2, x_3) = \left(\frac{4R^4}{R^4 + 2\left(t^2 + x_1^2 + x_2^2 + x_3^2\right) R^2 + \left(-t^2 + x_1^2 + x_2^2 + x_3^2\right)^2} \right) . \tag{29}$$

Therefore, in order to interpret (12) as a field theory on S^3 with a round metric, we should start with (24) on flat Minkowski metric, apply a conformal transformation, followed by a diffeomorphism with respect to the map (26). Luckily, gauge fields have conformal scaling dimension zero [39]. So F is invariant under conformal transformation. We therefore conclude that (12) is dual to $\mathcal{N} = 4$ theory with

$$F = qp\, d\tilde{\phi}_1 \wedge \tilde{\phi}_2 \tag{30}$$

with coordinates $\tilde{\phi}_i$ periodic under shift by $2\pi/p$.

To proceed further, we will view S^3 as T^2 parameterized by $(\tilde{\phi}_1, \tilde{\phi}_2)$, fibered over an interval I parameterized by $0 \leq \theta \leq \pi/2$. It is natural to express functions on S^3 in a basis

$$f(\theta, \tilde{\phi}_1, \tilde{\phi}_2) = g(\theta) e^{in_1\tilde{\phi}_1 + in_2\tilde{\phi}_2} . \tag{31}$$

The fact that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are periodic with respect to shift in $2\pi/p$ implies that n_1 and n_2 must be integer multiples of p. However, in the presence of a fractional flux [40, 41]

$$\int F = qp\frac{1}{p^2} = \frac{q}{p} , \tag{32}$$

the $p \times p$ degrees of freedom in the adjoint of SU(pN) splits into adjoints of SU(N) in a box whose size is larger by a factor of p [42, 43]. The non-commutative algebra of the $p \times p$ adjoint degrees of freedom are precisely isomorphic to the Moyal algebra with rational dimensionless non-commutativity parameter as was shown, e.g., in [44, 45]. These arguments are also reviewed in more detail in the appendix.

Since the argument is somewhat long winded, the outline of the argument is summarized in the flow chart diagram illustrated in figure 2. Our goal was to show that the Melvin twist of $AdS_5 \times S_5$ is the supergravity dual of NCSYM on S_3 with the non-commutative (ϕ_1, ϕ_2) coordinates, illustrated by a blue arrow in figure 2. We relied heavily on the SL$(2, Z)$ structure both on the field theory side and the supergravity side of the correspondence, as well as the rationality of the deformation parameter χ, to reformulate the theory in terms of an orbifold of $\mathcal{N} = 4$ theory. This allowed the duality from the open string/closed string perspective to be made most manifest. By following the chain of duality back to the original description, we derive the original duality of interest confirming [30]. This is the main result of this article.

The rationality of the deformation parameter χ and subsequent SL$(2, Z)$ transformation proved to be the powerful handle in defining these theories at the microscopic level.

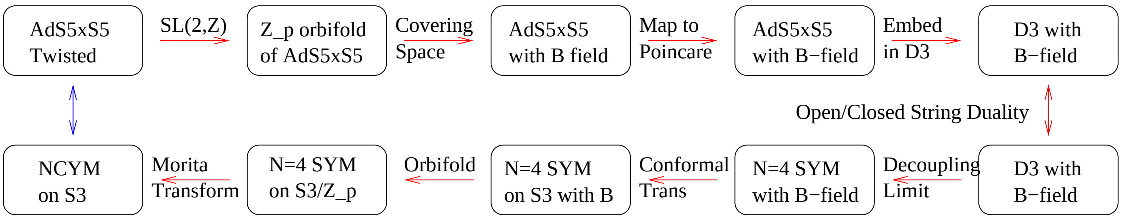

Figure 2: Schematic flowchart of the duality chain, demonstrating that the blue arrow in the far left is a consequence of the standard open/closed string duality correspondence on the far right.

It should be possible to formulate a microscopic formulation of Puff Field Theory along these lines as well [46].

It should be noted that strictly speaking, the deformation/orbifolding along ξ_i which we considered in this article breaks all supersymmetries (just as in the pure Melvin case of [18, 19]). What this means is that one expects the supergravity background to be unstable to decay, and for the field theory side to suffer from runaway vacua. However, the fact that the supergravity background considered in this article does satisfy the classical equation of motion implies, as was the case for various non-supersymmetric orbifolds [47], that the effects of instability are subleading in $1/N$ expansion. One could also imagine our analysis for ξ_1 and ξ_2 in $AdS_5 \times S_5$ which preserves some fraction of supersymmetry, such as choosing the ξ_1 to be along the Hopf fiber of S^3, and ξ_2 to be along the Hopf fiber of the S_3 of SO(4) $\in$ SO(6). More specifically, parameterize the metric of $AdS_5 \times S_5$ by coordinates

$$ds^2 = R^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{3(1)} + d\Omega_5^2\right] \tag{33}$$

where

$$d\Omega_5^2 = d\alpha^2 + \cos^2\alpha d\beta^2 + \sin^2\alpha d\Omega^2_{3(2)} \tag{34}$$

with

$$d\Omega^2_{3(i)} = d\Omega^2_{2(i)} + (d\phi_i + \mathcal{A}_i)^2, \quad d\Omega^2_{2(i)} = \frac{1}{4}(d\theta_i^2 + \sin^2\theta_i d\varphi_i^2), \quad \mathcal{A}_i = -\frac{1}{2}(1-\cos\theta_i)d\varphi_i \tag{35}$$

and set $\xi_i = \partial_{\phi_i}$. Performing a Melvin twist by the amount χ will give rise to a geometry

$$\begin{aligned} ds^2 \;=\; R^2 \Bigg[& -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho \left(d\Omega^2_{2(1)} + \frac{(d\phi_1 + \mathcal{A}_1)^2}{(1+\chi^2\lambda\sinh^2\rho\sin^2\alpha)}\right) + d\alpha^2 \\ & + \cos^2\alpha d\beta^2 + \sin^2\alpha\left(d\Omega^2_{2(2)} + \frac{(d\phi_2 + \mathcal{A}_2)^2}{(1+\chi^2\lambda\sinh^2\rho\sin^2\alpha)}\right)\Bigg] \end{aligned} \tag{36}$$

which is to be interpreted as an example of a dipole field theory [20, 21]. If the deformation parameter takes on a rational value $\chi = s/p$, this geometry can be mapped, via an SL$(2,Z)$ transformation, to $(AdS_5/Z_p) \times (S_5/Z_p)$ geometry with torsion

$$\begin{aligned} ds^2 \;=\; R^2 \Bigg[& -\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho \left(d\Omega^2_{2(1)} + \frac{1}{p^2}(d\phi_1 + \mathcal{A}_1)^2\right) + d\alpha^2 \\ & + \cos^2\alpha d\beta^2 + \sin^2\alpha\left(d\Omega^2_{2(2)} + \frac{1}{p^2}(d\phi_2 + \mathcal{A}_2)^2\right)\Bigg] \end{aligned} \tag{37}$$

preserving 1/4 of the original supersymmetry and should be stable. Other possible Killing vectors along which one can compactify and or twist preserving some fraction of supersymmetries can be found, e.g., in [48 – 51]. Along lines similar to [16], many of these constructions would constitute a laboratory for exploring issues of string theory in time dependent backgrounds.

Finally, let us consider the thermodynamics of the twisted $\mathrm{U}(1) \times \mathrm{U}(1) \in S^3$ theory from the supergravity point of view. Start with the Schwarzschild black hole solution [52]

$$ds^2 = -\left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right)} + r^2 d\Omega^2 \tag{38}$$

where $n = 4$ for the AdS_5, $w_n = \frac{16\pi G_N}{(n-1)Vol(S^{n-1})}$, and

$$d\Omega^2 = d\theta + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2 \,. \tag{39}$$

The period of t coordinate is given by

$$\beta = \frac{1}{T} = \frac{4\pi b^2 r_+}{4r_+^2 + 2b^2}, \qquad \frac{r_+^2}{b^2} + 1 - \frac{w_4 M}{r_+^2} = 0, \qquad r_+ = \text{horizon radius}\,, \tag{40}$$

and the boundary is conformal to $S_1 \times S_3$ with periods β and $R = b$, respectively.

One can then perform the χ deformation on this background, giving rise to a new background

$$\frac{ds^2}{\alpha'} = \sqrt{\lambda}\left[-\left(\cosh^2\rho - \frac{\mu}{\sinh^2\rho}\right) d\tau^2 + \frac{\cosh^2\rho}{\left(\cosh^2\rho - \frac{\mu}{\sinh^2\rho}\right)} d\rho^2 + \sinh^2\rho d\Sigma^2\right] \tag{41}$$

where we have changed coordinates to match the asymptotic behavior of (1)

$$t = R\tau, \qquad r^4 = R^4\sinh^4\rho = \alpha'^2\lambda\sinh^4\rho \tag{42}$$

and

$$d\Sigma^2 = d\theta^2 + \frac{\sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2}{1 + \lambda\chi^2\cos^2\theta\sin^2\theta\sinh^4\rho} \tag{43}$$

$$\mu = \frac{w_n M}{R^2} = \pi^4 R^4 T^4 + (\text{terms subleading in } 1/TR) \tag{44}$$

Just as in the undeformed case, the use of Schwarzschild black hole solution suffers from the Hawking-Page transition at low temperatures, but for $T > 1/R$, it follows from the standard reasoning that the entropy

$$S(T) = \frac{\pi^2}{2} N^2 V T^3\,, \tag{45}$$

being proportional to the area of the horizon in the Einstein frame, is unaffected by χ.

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A. Specturm and interaction of fluctuating fields on a torus with a 't Hooft flux

In this appendix, we show explicitly that U(p) gauge theory on a torus of size $L \times L$ with fractional flux q/p is equivalent to a non-commutative U(1) gauge theory with non-commutativity parameter $\theta = 2\pi s/p \times (pL)^2$ on a torus of size $pL \times pL$. This is a standard foliation argument of non-commutative torus [44, 45] but we will follow the notation and conventions of [43].

Consider U(p) gauge theory on box size $L \times L$ with fractional flux q/p. Convenient gauge is

$$\begin{aligned} A_1^0 &= 0 \\ A_2^0 &= F_0 x_1 I + \frac{2\pi}{L_2}\mathrm{Diag}(0, 1/p, \ldots, (p-1)/p) \end{aligned} \tag{46}$$

where

$$F_0 = \frac{2\pi}{L_1 L_2}\frac{q}{p}.$$

Adjoint scalars in such a background will satisfy the boundary condition

$$\begin{aligned} \Phi(x_1 + L_1, x_2) &= e^{2\pi i (x_2/L_2)T} V^q \Phi(x_1, x_2) V^{-q} e^{-2\pi i (x_2/L_2)T} \\ \Phi(x_1, x_2 + L_2) &= \Phi(x_1, x_2) \end{aligned} \tag{47}$$

Treating the action to the quadratic order, the plane wave solution with this boundary condition is

$$\delta\Phi_{m_1,m_2,r}(x_1, x_2) = \varphi_{m_1,m_2,r}\Lambda_{m_1,m_2,r}e^{2\pi i(m_1 x_1/L_1 + m_2 x_2/L_2)}$$

where

$$m_1 \in \mathbf{Z}/p, \qquad m_2 \in \mathbf{Z}, \qquad r = 0 \ldots p-1 \tag{48}$$

and

$$\Lambda_{m_1,m_2,r} = \mathrm{Diag}\{1, \omega, \omega^2, \ldots, \omega^{p-1}\} \cdot \left(\begin{array}{cccccc} e^{-2\pi i x_2/L_2} & & & & & \\ & \ddots & & & & \\ & & e^{-2\pi i x_2/L_2} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} p-r \end{array} \cdot V^{-r} \tag{49}$$

where $\omega = e^{2\pi i m_1 s}$ for $qs \equiv 1 \pmod{p}$.

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The energy and momentum carried by these modes (see (15)-(17) of [43]) are

$$E^2 = k_1^2 + k_2^2, \qquad k_1 = \frac{2\pi m_1}{L_1}, \qquad k_2 = \frac{2\pi}{L}\left(m_2 - \frac{r}{p}\right) \tag{50}$$

which in light of (48) is identical to that of a single degree of freedom in a box of size pL, rather than p^2 degrees of freedom in a box of size L.

Let us define an algebra for the $\varphi(m_1, m_2, r)$ that is homomorphic to the algebra of $\Phi_{m_1,m_2,r}(x_1, x_2)$. In other words, we want

$$\Phi[\varphi_{k_1,k_2}(x_1,x_2) * \varphi_{k_1',k_2'}(x_1,x_2)] = \Phi[\varphi_{k_1,k_2}(x_1,x_2)] \cdot \Phi[\varphi_{k_1',k_2'}(x_1,x_2)] \tag{51}$$

where

$$\varphi_{k_1,k_2}(x_1,x_2) = \varphi_{k_1,k_2} e^{ik_1x_1 + ik_2x_2} \tag{52}$$

We find

$$\varphi_{k_1,k_2}(x_1,x_2) * \varphi_{k_1',k_2'}(x_1,x_2) = e^{ik_1\theta k_2'}(\varphi_{k_1+k_1',k_2+k_2'}(x_1,x_2) \tag{53}$$

follows from the basic fact that

$$\Lambda_r \Lambda_{r'} = \omega'^{-r} \Lambda_{r+r'} \tag{54}$$

To see this, note that the phase factor

$$\omega'^{-r} = e^{-2\pi i m_1' s r} = e^{-2\pi i m_1' s(r - pm_2)} = e^{\frac{i(pL)^2 s}{2\pi p} k_1' k_2} \tag{55}$$

from which we read off that

$$\theta = \frac{s}{p} \cdot \frac{(pL)^2}{2\pi} \,. \tag{56}$$

We see that this is precisely the non-commutativity parameter one expects to find by starting with q units of flux in a U(p) theory and acting by an SL($2, Z$) element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & -q \\ s & r \end{pmatrix} \tag{57}$$

which is the inverse of (8), and which according to (1.9) of [53] maps the theory to a U(1) theory with no 't Hooft flux. The condition $qs = 1 \mod p$ is precisely the SL($2, Z$) condition $pr + sq = 1$.

Now, this is not quite the Moyal product, but it can be shown to be isomorphic to it. Under the map

$$\varphi_{k_1,k_2}(x_1,x_2) = e^{-ik_1\theta k_2/2} \tilde{\varphi}_{k_1,k_2}(x_1,x_2) \tag{58}$$

the algebra becomes

$$\tilde{\varphi}_{k_1,k_2}(x_1,x_2) * \tilde{\varphi}_{k_1',k_2'}(x_1,x_2) = e^{i(k_1\theta k_2' - k_2\theta k_1')/2} \tilde{\varphi}_{k_1+k_1',k_2+k_2'}(x_1,x_2) \,. \tag{59}$$

The same argument, applied to the T^2 fiber of S^3, gives rise to an algebra

$$f(\theta,\phi_1,\phi_2) * g(\theta,\phi_1,\phi_2) = e^{2\pi i \Theta_{ij}\partial_{\phi_i}\partial_{\phi_j'}/2} f(\theta,\phi_1,\phi_2) g(\theta,\phi_1',\phi_2')\Big|_{\phi_i=\phi_i'} \tag{60}$$

with

$$\Theta_{12} = -\Theta_{21} = \frac{s}{p} \tag{61}$$

which is the non-commutative deformation (6) along (ϕ_1, ϕ_2) coordinates of S^3.

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